PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 12

PRACE MATEMATYCZNE II, 1972

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Some local properties of re a_3, re a_4 and re a_5 in the class S

INTRODUCTION. Let S be the clas of all holomorphic and univalent functions $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ in the unit disc $|z| < 1$. Each function $f(z)$ maps conformally the disc $|z| < 1$ onto a domain D which contains the orgin 0 of the coordinate system and $w = 1/f(z)$ maps $|z| < 1$ onto a domain G, where G is the exterior of a continuum E of capacity 1. The origin $w = 0$ is contained in E. To each function $f(z) \in S$ corresponds in one-to-one way a continuum E, $\operatorname{cap} E = 1$, and the coordinate system with the origin $0 \in E$. When E is fixed, $\operatorname{cap} E = 1$, $0 \in E$, then the rotation of the coordinate system around 0 gives a new function in S but the absolute values of the coefficients a_n, $n = 1, 2, \dots$ remain unchanged.

Let $\eta_1, \eta_2, \dots, \eta_n$ be the n^{th} extremal system of points in E i.e. a system of n points in E such that $\Pi\,|\eta_j - \eta_k| = \sup\limits_{(w_j) \in E} \Pi\,|w_j - w_k|$ $1 \leqslant j < k \leqslant n$ F. LEJA proved in [1] the existence of the following limits

$$\lim\, (\eta_1^k + \eta_2^k + \dots + \eta_n^k)/n = s_k, \quad k = 1, 2, \dots$$

The point $s_1 = 0\bar{0}$ is the the center of gravity of the natural mass distribution on E and its position relative to E remains unchanged after rotation or translation of the coordinate system.

The aim of this note is to prove some simple inequalities concerning re a_3, re a_4, and re a_5.

AUXILIARY FORMULAS. Let us consider the set of all continuums E of capacity 1, situated so that center of gravity $\bar{0}$ is the common point for all E. It is known that all E lie inside the disc K of radius 2 centerd at the point $\bar{0}$. Among all E under consideration there are segments of length 4 which have their endpoints on the circumference of K, all other E have a positive distance from the boundary of K.

F. LEJA [1] gave formulas which express the coefficients a_n $n = 2, 3, \ldots$ of $f(z) \in S$ as polynomials in $s_1, s_2, \ldots$ If one computes „the moments" s_k relative to the point $\overline{0}$ instead of to the origin 0 one obtains

$$s_k = s_k(\overline{0}) + \binom{k}{1} s_{k-1}(\overline{0}) s_1 + \binom{k}{2} s_{k-2}(\overline{0}) s_1^2 + \ldots + s_1^k$$

The formulas given by F. Leja are the following

$$a_2 = -s_1, \; a_3 = a_2^2 - s_2(0)/2, \; a_4 = a_2^3 - a_2 s_2(\overline{0}) - s_3(\overline{0})/3,$$

$$a_5 = a_2^4 - 3a_2^2 s_2(\overline{0})/2 - 2a_2 s_3(\overline{0})/3 - s_4(\overline{0})/4 + 5s_2^2(\overline{0})/8.$$

As the rotation of the coordinate system does not change the modulus of a_n, $n = 1,2,\ldots$ we can choose it so that the real axis has the direction of $\overline{00}$ i.e. $a_2 = \overline{00} > 0$.

For the Koebe function $z/(1-z)^2$ is

$$a_2 = 2, \; s_2(0) = 2, \; s_3(\overline{0}) = 0, \; s_4(\overline{0}) = 6.$$

On the other hand (see [2]) for all $f(z) \in S$ is

$$|a_2| \leqslant 2, \; |s_2(\overline{0})| \leqslant 2, \; |s_4(\overline{0})| \leqslant 6$$

the equality holds only for $f(z) = z/(1 - ze^{i\vartheta})^2$. In the general case

$$\begin{aligned} & a_2 = 2 - \varepsilon + i\varepsilon_1, \; \varepsilon \geqslant 0 \\ (1) \qquad & s_2(0) = 2 - \delta + i\delta_1, \; \delta \geqslant 0 \\ & s_4(\overline{0}) = 6 - \eta + i\eta_1, \; \eta \geqslant 0 \end{aligned}$$

and

$$\text{re } a_3 = 3 - 4\varepsilon + \frac{1}{2}\delta + \varepsilon^2 - \varepsilon_1^2,$$

$$(2) \; \text{re } a_4 = 4 - 10\varepsilon + 2\delta - \text{re } s_3(\overline{0})/3 + 6\varepsilon^2 - 6\varepsilon_1^2 - \varepsilon^3 + 3\varepsilon_1^2 + \varepsilon_1\delta_1 - \varepsilon\delta,$$

$$\begin{aligned} \text{re } a_5 = {} & 5 - 20\varepsilon + 7\delta/2 + \eta/4 - 4 \text{ re } s_3(\overline{0})/3 - 5\delta_1^2/8 + 5\delta^2/8 - 21\varepsilon_1^2 + \\ & + 21\varepsilon^2 + 2\varepsilon_1 \text{ im } s_3(\overline{0})/3 + 2\varepsilon \text{ re } s_3(\overline{0})/3 - 8\varepsilon^3 + \varepsilon^4 + \varepsilon_1^4 + \\ & + 3\varepsilon^2\delta/2 + 6\varepsilon_1\delta_1 + 24\varepsilon\varepsilon_1^2 - 6\varepsilon^2\varepsilon_1^2 - 3\varepsilon_1^2\delta/2 - 3\varepsilon\varepsilon_1\delta_1 - 6\varepsilon\delta. \end{aligned}$$

Let us consider the class S of all those functions $f(z) \in S$ to which corresponds as a continuum E the segment I of length 4 and the middle at 0. Each function $f(z) \in S$ has the form

$$f(z) = z - e^{i\vartheta}\varrho z^2 + e^{2i\vartheta}(-1 + \varrho^2) z^3 + e^{3i\vartheta}(2\varrho - \varrho^3) z^4 + \ldots$$

where $0 \leqslant \vartheta \leqslant 2\pi$, $-2 \leqslant \varrho \leqslant 2$. When $a_2 = -e^{i\vartheta}$ is > 0, then $\vartheta = \pi$, $\varrho \in [0,2]$ and

$$f(z) = z + \varrho z^2 + (-1 + \varrho^2)z^3 + (-2\varrho + \varrho^3)z^4 + (1 - 3\varrho^3 + \varrho^4)z^5 + \ldots$$

Therefore the second coefficient of all $f(z) \in S$ with positive a_2 has the value $\varrho = 2 - \varepsilon$, $\varepsilon \geqslant 0$.

THEOREM 1. *Let us consider all functions $f(z) \in S$ with the same second coefficient $a_2 > 0$, $a_2 < 2$. Then the smallest value of re a_3 is assumed by a function $f(z) \in \bar{S}$. For $a_2 > 0$ sufficiently close to 2 the smallest value of re a_4 and re a_5 are assumend by functions in $\bar{S}$*

Proof. 1°. From (2) re $a_3 - 3 = -4\varepsilon - \frac{1}{2}\delta + \varepsilon^2$. Let us denote by $\bar{a}_3$ the third coefficient of a function in $\bar{S}$, then re $\bar{a}_3 - 3 = -4\varepsilon + \varepsilon^2$. Therefore re a_3 − re $\bar{a}_3 = \frac{1}{2}\delta > 0$.

2°. From (2) re a_4 − re $\bar{a}_4 = 2\delta$ − re $a_3(\bar{0})/3 - \varepsilon\delta$. One of Grunsky inequalities (see [2]) has the form

$$(3) \qquad \left| \frac{s_2(\bar{0})}{2}\xi_1^2 + \frac{4}{3}s_3(\bar{0})\xi_1\xi_2 + [s_4(\bar{0}) - s_2^2(\bar{0})]\xi_2^2 \right| \leqslant |\xi_1|^2 + 2|\xi_2|^2$$

where ξ_1 and ξ_2 are arbitarary numbers, and the equality holds only for the Koebe function. Taking the real parts of both sides for $\xi_1 = 2$ and $\xi_2 = \frac{1}{2}$ we obtain (using previous notations)

$$-2\delta + \frac{4}{3}\,\mathrm{re}\, s_3(\bar{0}) - \frac{\eta}{4} + \delta - \frac{\delta^2}{4} + \frac{\delta_1^2}{4} < 0.$$

Hence

$$(4) \qquad -\frac{\mathrm{re}\, s_3(\bar{0})}{3} > -\frac{\eta}{16} - \frac{\delta}{4} + \frac{\delta_1^2}{16} - \frac{\delta^2}{16}.$$

On the other hand it was proved in [3]

$$\left| \frac{3}{4}s_2^2(\bar{0}) - \frac{s_4(\bar{0})}{4} \right| \leqslant 3/2 \quad \text{for all } f(z) \in S.$$

Therefore

$$(5) \qquad -3\delta - \frac{3}{4}\delta_1^2 + \frac{3}{4}\delta^2 + \frac{\eta}{4} < 0 \Rightarrow \frac{\eta}{4} < 3\delta + \frac{3}{4}\delta_1^2 - \frac{3}{4}\delta^2.$$

From (4) and (5)

$$-\frac{\mathrm{re}\, s_3(\bar{0})}{3} > -\delta - \frac{\delta_1^2}{8} + \frac{\delta^2}{8}.$$

Hence

$$\mathrm{re}\, a_4 - \mathrm{re}\, \bar{a}_4 > \delta - \varepsilon\delta + \frac{\delta^2}{8} - \frac{\delta_1^2}{8}.$$

But

$$\delta_1^2 + (2-\delta)^2 = |s_2(\bar{0})|^2 = (2-c_2)^2 > c_2 > 0$$

for all $f(z) \in S$ different from the Koebe function. From the last formula follows

$$\frac{\delta_1^2}{8} = \frac{1}{2}(\delta - c_2) + \frac{c_2^2 - \delta^2}{8} < \frac{1}{2}(\delta - c_2), \quad \delta \geqslant c_2 > 0.$$

Hence

$$\operatorname{re} a_4 - \operatorname{re} \bar{a}_4 > \delta - \varepsilon\delta + \frac{\delta^2}{8} - \frac{\delta}{2} + \frac{c_2}{2} > \delta\left(\frac{1}{2} - \varepsilon\right) > 0$$

for sufficiently smal $\varepsilon > 0$.

3°. We have

$$\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 = \frac{7}{2}\delta + \frac{\eta}{4} - \frac{4}{3}\operatorname{re} s_3(\bar{0}) - \frac{5}{8}\delta_1^2 + \frac{5}{8}\delta^2 +$$

$$+ \frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) + \frac{3}{2}\varepsilon^2\delta - 6\varepsilon\delta.$$

According to (4)

$$\frac{\eta}{4} - \frac{4}{3}\operatorname{re} s_4(\bar{0}) > -\delta - \frac{\delta^2}{4} + \frac{\delta_1^2}{4}.$$

Therefore

$$\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > \frac{5}{2}\delta - \frac{3}{8}\delta_1^2 + \frac{3}{8}\delta^2 + \frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) + \frac{3}{2}\varepsilon^2\delta - 6\varepsilon\delta$$

$$\delta + \frac{3}{2}c_2 + \frac{3}{8}\delta^2 + \frac{3}{2}\varepsilon^2\delta - 6\varepsilon\delta + \frac{2}{3}\operatorname{re} s_3(\bar{0}) > \delta[1 - 6\varepsilon] + \frac{2}{3}\varepsilon \operatorname{re} s_3(0)$$

If we put in (3) $\xi_1 = -2, \quad \xi_2 = \frac{1}{2}$ we obtain

$$\frac{\operatorname{re} s_3(\bar{0})}{3} > -\frac{\eta}{16} - \frac{\delta}{4} - \frac{\delta^2}{16} + \frac{\delta_1^2}{16}$$

and multiplying by $2\varepsilon > 0$

$$\frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) > -\frac{\eta}{8}\varepsilon - \frac{\delta\varepsilon}{2} - \frac{\delta^2\varepsilon}{8} + \frac{\delta_1^2\varepsilon}{8}.$$

Using (5)

$$\frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) > -2\delta\varepsilon - \frac{\delta_1^2\varepsilon}{4} + \frac{\delta^2\varepsilon}{4} > -3\varepsilon\delta + \varepsilon c_2 + \frac{\delta^2\varepsilon}{4}$$

whence

$$\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > \delta(1 - 9\varepsilon)$$

and for sufficently small $\varepsilon > 0$

$$\operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > 0.$$

THEOREM 1*. *Put* $a_2 = (2 - c)\cos\alpha$. *Then for sufficiently small* $c > 0$ *and* $|\alpha| \geqslant 0$

$$\operatorname{re} a_3 - \operatorname{re} \bar{a}_3 > 0, \quad \operatorname{re} a_4 - \operatorname{re} \bar{a}_4 > 0, \quad \operatorname{re} a_5 - \operatorname{re} \bar{a}_5 > 0.$$

THEOREM 2. *If* $a_2 = 2 - \varepsilon$, $\varepsilon > 0$, *then*

$$\operatorname{re}[a_5 + s_4(\bar{0})/4] < 6\frac{1}{2}$$

for all sufficiently small $\varepsilon > 0$.

Proof. From (2) follows $\operatorname{re}\left[a_5 + \frac{s_4(\bar{0})}{4}\right] = 6\frac{1}{2} - 20\varepsilon + \frac{7}{2}\delta -$

$$-\frac{4}{3}\operatorname{re} s_3(\bar{0}) - \frac{5}{8}\delta_1^2 + \frac{5}{8}\delta^2 + 21\varepsilon^2 + \frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) - 8\varepsilon^3 + \varepsilon^4 - \frac{3}{2}\varepsilon^2\delta - 6\varepsilon\delta$$

and

$$\frac{7}{4}(\operatorname{re} a_4 - 4) = -\frac{70}{4}\varepsilon + \frac{7}{2}\delta - \frac{7}{12}\operatorname{re} s_3(\bar{0}) + \frac{21}{2}\varepsilon^2 - \frac{7}{4}\varepsilon^3 - \frac{7}{4}\varepsilon\delta.$$

Hence

$$\operatorname{re}\left[a_5 + \frac{s_4(\bar{0})}{4}\right] = \frac{7}{4}(\operatorname{re} a_4 - 4) + 6\frac{1}{2} - \frac{5}{2}\varepsilon - \frac{3}{4}\operatorname{re} s_3(\bar{0}) - \frac{5}{8}\delta_1^2 +$$
$$+\frac{5}{8}\delta^2 + \frac{21}{2}\varepsilon^2 + \frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) - \frac{25}{4}\varepsilon^3 + \varepsilon^4 + \frac{3}{2}\varepsilon^2\delta - \frac{17}{4}\varepsilon\delta.$$

From the inequality

$$\left|\frac{a_2{}^3}{12} - \frac{s_3(\bar{\bar{0}})}{3}\right| \leqslant \frac{2}{3}, \quad f(z) \in S$$

obtained by M. Schiffer follows

$$-\varepsilon - \frac{\operatorname{re} s_3(\bar{0})}{3} - \frac{\varepsilon_1^2}{2} + \frac{\varepsilon\,\varepsilon_1^2}{4} + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{12} < 0.$$

Therefore ($\varepsilon_1 = 0$)

$$\frac{3}{4}\operatorname{re} s_3(\bar{0}) < \frac{9}{4}\varepsilon - \frac{9}{8}\varepsilon^2 + \frac{3}{16}\varepsilon^3$$

and

$$\operatorname{re}\left[a_5 + \frac{s_4(\bar{0})}{4}\right] < 6\frac{1}{2} - \frac{\varepsilon}{4} - \frac{5}{8}\delta_1^2 + \frac{5}{8}\delta^2 + \frac{2}{3}\varepsilon \operatorname{re} s_3(\bar{0}) +$$
$$+\frac{73}{8}\varepsilon^2 - \frac{97}{16}\varepsilon^3 + \varepsilon^4 + \frac{3}{2}\varepsilon^2\delta - \frac{17}{4}\varepsilon\delta < 6\frac{1}{2}$$

for sufficiently small

$$\varepsilon > 0.$$

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WŁASNOŚCI LOKALNE re a_3, re a_4 I re a_5 W KLASIE S

Streszczenie

Oznaczmy przez S klasę funkcji $f(z) = z+a_2z^2+\ldots$ holomorficznych i różnowartościowych w kole $|z| < 1$. Każda funkcja $1/f(z)$ odwzorowuje koło jednostkowe na obszar G, którego uzupełnienie do całej płaszczyzny jest kontinuum E o pojemności logarytmicznej 1. Każdej funkcji klasy S odpowiada pewne kontinuum E, poj. $E = 1$ oraz układ odniesienia o początku 0 leżącym na E.

F. Leja podał wzory na współczynniki a_n funkcji $f(z) \in S$ przy pomocy średnich s_k, $k = 1, 2, \ldots$. Jeżeli przez $\bar{S}$ oznaczyć podklasę klasy S składającą się z tych funkcji $f(z)$, którym odpowiada jako kontinuum E odcinek o długości 4, to zachodzą następujące nierówności

$$\operatorname{re} a_3 > \operatorname{re} \bar{a}_3, \quad \operatorname{re} a_4 > \operatorname{re} \bar{a}_4, \quad \operatorname{re} a_5 > \operatorname{re} \bar{a}_5$$

dla wszystkich funkcji o tym samym współczynniku $a_2 > 0$, dostatecznie bliskiemu 2.

Podobnie wykazuje się, że dla $f(z) \in S$ gdy $a_2 > 0$ jest dostatecznie bliskie 2, zachodzi nierówność $\operatorname{re}\left[a_5 + \frac{\bar{s}_4(0)}{4}\right] < 6\frac{1}{2}$

Oddano do Redakcji 3. IV. 1970 r.